N07/5/MATHL/HP2/ENG/TZ0/XX

88077202

MATHEMATICS
HIGHER LEVEL
PAPER 2

Tuesday 6 November 2007 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 20]*

(a) A curve is defined by the implicit equation $2xy + 6x^2 - 3y^2 = 6$.

Show that the tangent at the point A with coordinates $\left(1, \frac{2}{3}\right)$ has gradient $\frac{20}{3}$. *[6 marks]*

(b) The line $x = 1$ cuts the curve at point A, with coordinates $\left(1, \frac{2}{3}\right)$, and at point B.

Find, in the form $\boldsymbol{r} = \begin{pmatrix} a \\ b \end{pmatrix} + s\begin{pmatrix} c \\ d \end{pmatrix}$

(i) the equation of the tangent at A;

(ii) the equation of the normal at B. *[10 marks]*

(c) Find the acute angle between the tangent at A and the normal at B. *[4 marks]*

2. *[Total mark: 22]*

Part A *[Maximum mark: 13]*

(a) The function f is defined by $f(x) = (x+2)^2 - 3$.
The function g is defined by $g(x) = ax + b$, where a and b are constants.

Find the value of a, $a > 0$ and the corresponding value of b, such that

$$f\left(g(x)\right) = 4x^2 + 6x - \frac{3}{4}.$$

[8 marks]

(b) The functions h and k are defined by $h(x) = 5x + 2$ and $k(x) = cx^2 - x + 2$ respectively. Find the value of c such that $h\left(k(x)\right) = 0$ has equal roots. *[5 marks]*

Part B *[Maximum mark: 9]*

(a) Express the complex number $1+\mathrm{i}$ in the form $\sqrt{a}\mathrm{e}^{\mathrm{i}\frac{\pi}{b}}$, where $a, b \in \mathbb{Z}^+$. *[2 marks]*

(b) Using the result from (a), show that $\left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^n$, where $n \in \mathbb{Z}$, has only eight distinct values. *[5 marks]*

(c) **Hence** solve the equation $z^8 - 1 = 0$. *[2 marks]*

3. *[Total mark: 30]*

Part A *[Maximum mark: 18]*

On a particular road, serious accidents occur at an average rate of two per week and can be modelled using a Poisson distribution.

(a) (i) What is the probability of at least eight serious accidents occurring during a particular four-week period?

(ii) Assume that a year consists of thirteen periods of four weeks. Find the probability that in a particular year, there are more than nine four-week periods in which at least eight serious accidents occur. *[10 marks]*

(b) Given that the probability of at least one serious accident occurring in a period of n weeks is greater than 0.99, find the least possible value of n, $n \in \mathbb{Z}^+$. *[8 marks]*

Part B *[Maximum mark: 12]*

A continuous random variable X has probability density function defined by

$$f(x) = \begin{cases} \dfrac{c}{4+x^2}, & \text{for } -\dfrac{2}{\sqrt{3}} \le x \le 2\sqrt{3} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the **exact** value of the constant c in terms of π. *[5 marks]*

(b) Sketch the graph of $f(x)$ and hence state the mode of the distribution. *[3 marks]*

(c) Find the **exact** value of $\mathrm{E}(X)$. *[4 marks]*

4. *[Maximum mark: 25]*

The function f is defined by $f(x) = \operatorname{cosec} x + \tan 2x$.

(a) Sketch the graph of f for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Hence state

(i) the x-intercepts;

(ii) the equations of the asymptotes;

(iii) the coordinates of the maximum and minimum points. *[8 marks]*

(b) Show that the roots of $f(x) = 0$ satisfy the equation $2\cos^3 x - 2\cos^2 x - 2\cos x + 1 = 0$. *[5 marks]*

(c) Show that the x-coordinates of the maximum and minimum points on the curve satisfy the equation $4\cos^5 x - 4\cos^3 x + 2\cos^2 x + \cos x - 2 = 0$. *[8 marks]*

(d) Show that $f(\pi - x) + f(\pi + x) = 0$. *[4 marks]*

5. *[Total mark: 23]*

Part A *[Maximum mark: 11]*

The acceleration in m s^{-2} of a particle moving in a straight line at time t seconds, $t > 0$, is given by the formula $a = -\frac{1}{(1+t)^2}$. When $t = 1$, the velocity is $8\ \text{m s}^{-1}$.

(a) Find the velocity when $t = 3$. *[6 marks]*

(b) Find the limit of the velocity as $t \to \infty$. *[1 mark]*

(c) Find the exact distance travelled between $t = 1$ and $t = 3$. *[4 marks]*

Part B *[Maximum mark: 12]*

Given that $y = x\mathrm{e}^{-x}$,

(a) find $\frac{\mathrm{d}y}{\mathrm{d}x}$; *[2 marks]*

(b) use mathematical induction to prove that, for $n \in \mathbb{Z}^+$, $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n+1}\mathrm{e}^{-x}(n-x)$. *[10 marks]*